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$$=-\frac{1}{c}\frac{\sinh(x/c)\ \cosh(x/c)}{[1+\cosh^2(x/c)]^2}.$$

$$\frac{d^2y}{ds^2}=-\tfrac{1}{2}(1+\cosh^2\frac{x}{c})^{-\frac{3}{2}}.2\cosh\frac{x}{c}\sinh\frac{x}{c}.\frac{1}{c}.\ (1+\cosh^2\frac{x}{c})^{-\frac{1}{2}}.\cosh\frac{x}{c}$$

$$+\sinh\frac{x}{c}.\frac{1}{c}(1+\cos^2\frac{x}{c})^{-\frac{1}{2}}.(1+\cosh^2\frac{x}{c})^{-\frac{1}{2}}=\frac{1}{c}\frac{\sinh(x/c)}{[1+\cosh^2(x/c)]^2}.$$

Also solved by H. C. Feemster, G. B. M. Zerr, V. M. Spunar, and J. Scheffer.

278. Proposed by S. A. COREY, Hiteman, Iowa.

If C be Euler's constant, .577,215,664,9... and if B_1, B_2, B_3, etc., be Bernoulli's numbers, $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, etc., prove that

$$C=\tfrac{1}{2}+\frac{B_1}{2}-\frac{B_2}{4}+\frac{B_3}{6}-\frac{B_4}{8}+\ldots-(1)^m\frac{B_m}{2m}+\ldots.$$

I. Solution by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Euler's constant C may be presented under various forms from among which an elementary one may be the following:

$$C=\lim_{n\doteq\infty}[1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots\frac{1}{n}-\log n]=.577,215,664,9\ldots$$

By Taylor's Theorem, we have

$$f(a+h)-f(a)=hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\ldots+\frac{h^n}{n!}f^{(n)}(a)+\ldots$$

Change a successively into $a+h$, $a+2h$, ..., $a+(n-1)h$ and add, then if we put x for $(a+nh)$, we obtain the following result:

$$f(x)-f(a)=h\,\Sigma f'(x)+\frac{h^2}{2!}\Sigma f''(x)+\frac{h^3}{3!}\Sigma f'''(x)+\ldots+\frac{h^3}{n!}\Sigma f^{(n)}(x)+\ldots$$

where $\Sigma f'(x)=f'(x)+f'(a+h)+\ldots$, $\Sigma f''(x)=f''(a)+f''(a+h)+\ldots$, etc.

Let $\phi(x)=f'(x)$. Then

$$\int_a^{a+nh}\phi(x)dx=h\,\Sigma\phi(x)+\frac{h^2}{2!}\Sigma\phi'(x)+\frac{h^3}{3!}\Sigma\phi''(x)+\ldots+\frac{h^n}{n!}\Sigma\phi^{(n-1)}(x)+\ldots.$$

$$\therefore \Sigma\, \phi\, (x) = \frac{1}{h}\int_a^{a+nh} \phi(x)\,dx - \frac{h}{2\,!}\,\Sigma\,\phi'(x) - \frac{h^2}{3\,!}\,\Sigma\,\phi''(x) - \ldots$$

$$-\frac{h^n}{(n+1)\,!}\,\Sigma\,\phi^{(n)}(x) - \ldots (1).$$

Similarly, $\Sigma\,\phi'(x) = \frac{1}{h}[\phi(x) - \phi(a)] - \frac{h}{2\,!}\,\Sigma\,\phi''(x) - \frac{h^2}{3\,!}\,\Sigma\,\phi'''(x) - \ldots$

$$-\frac{h^n}{(n+1)\,!}\,\phi^{(n+1)}(x) - \ldots (2).$$

Now from the series (1) we may eliminate $\Sigma\,\phi'(x)$, $\Sigma\,\phi''(x)$, ..., by the aid of (2), (3), ... In order to do it multiply (2) by $A_0 h$, (3) by $A_1 h^2$, ..., then add the results and determine A_0, A_1, A_2, ..., by equations

$$A_0 + \frac{1}{2\,!} = 0.$$

$$A_1 + \frac{A_0}{2\,!} + \frac{1}{3\,!} = 0.$$

$$A_2 + \frac{A_1}{2\,!} + \frac{A_0}{3\,!} + \frac{1}{4\,!} = 0.$$

. . . . .

$$\therefore \Sigma\,\phi(x) = \frac{1}{h}\int_a^{a+nh} \phi(x)\,dx + A_0[\phi(x) - \phi(a)] + A_1[\phi'(x) - \phi'(a)]h$$

$$+ A_2[\phi''(x) - \phi''(a)]h^2 + \ldots$$

As it has been shown that $\Sigma\,\phi(x)$ can be put in this form, where A_0, A_1, A_2, ..., A_n, ... are numerical quantities independent of the variable x and of the function $\phi(x)$, viz:

$\Sigma\,\phi(x) = \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{B_1}{2\,!}x^2 - \frac{B_2}{4\,!}x^4 + \ldots\ (-1)^{m+1}\frac{B_m}{2m\,!} + \ldots$, where

$A_m = (-1)^{m+1}\frac{B_m}{2m\,!}$, and B_1, B_2, B_3, ..., B_m are the well known Bernoulli's numbers ($A_0 = -\frac{1}{2}$).

$$\therefore \Sigma\, \phi(x) = \frac{1}{h}\int_a^x \phi(x)\,dx - \tfrac{1}{2}[\phi(x)-\phi(a)] + \frac{B_1}{2\,!}[\phi'(x)-\phi'(a)]h$$

$$- \frac{B_2}{4\,!}[\phi'''(x)-\phi'''(a)]h^3 + \ldots$$

Also,

$$\therefore \Sigma\, \phi(x) = C + \frac{1}{h}\int \phi(x)\,dx - \tfrac{1}{2}\phi(x) + \frac{B_1}{2}\phi'(x)\frac{h}{1\,!} - \frac{B_2}{4}\phi'''(x)\,\frac{h^3}{3\,!} + \ldots$$

$$\ldots(-1)^{m+1}\frac{B_m}{2_m}\phi^{(2m-1)}(x)\,\frac{h^{2m-1}}{(2m-1)\,!} + \ldots (M),$$

where C represents a series of terms independent of x.

Taking $\phi(x)=1/x$ and $h=1$ then we get, adding $(1/x)$ to both sides,

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{x} = C + \log x + \frac{1}{2x} - \frac{1}{12x^2} + \ldots$$

Hence, by making $x=\infty$ we infer that in this example C is Euler's constant (according to our definition).

But on the other hand in (M),

$$C = -\frac{1}{h}\int \phi(a)\,da + \tfrac{1}{2}\,\phi\,(a) + \frac{B_1}{2}\phi'(a)\frac{h}{1\,!} - \frac{B_2}{4}\;\phi'''(a)\;\frac{h_3}{3\,!} +$$

$$\ldots(-1)^{m+1}\,\frac{B_m}{2m}\phi^{(2m-1)}(a)\;\frac{h^{2m-1}}{(2m-1)\,!}\ldots (N).$$

Hence, put $\phi(a)=1/a$, $h=1$ (as above), and then $a=1$.

$$\therefore C \equiv \tfrac{1}{2} + \frac{B_1}{2} - \frac{B_2}{4} + \frac{B_3}{6} - \ldots (-1)^{m+1}\frac{B_m}{2m}\ldots \quad \text{Q. E. D.}$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Taking $a=1$, $x=m$ in the development in problem 237, Calculus, we get

$$\log(1+m) = 0 + \tfrac{1}{2}\left[\frac{1}{1+m} + 1 + 2\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{m}\right)\right]$$

$$+ \frac{B_1}{2!}\left[\frac{1}{(1+m)^2} - 1\right] - \frac{B_2}{4!}\left[\frac{1}{(1+m)^4} - 3!\right] + \text{etc.}$$

Let m approach ∞. Then

$$\lim_{n \doteq \infty} \log n = \left[\tfrac{1}{2} + \sum_{p=2}^{p=\infty} \frac{1}{p}\right] - \left[\frac{B_1}{2} - \frac{B_2}{4} + \frac{B_3}{6} - \ldots\right]$$

But $\displaystyle\lim_{n \doteq \infty} \log n - \sum_{p=2}^{p=\infty} \frac{1}{p} = -C+1.$

$$\therefore -C+1 = \tfrac{1}{2} - \frac{B_1}{2} + \frac{B_2}{4} - \frac{B_3}{6} + \frac{B_4}{8} - \ldots$$

$$\therefore C = \tfrac{1}{2} + \frac{B_1}{2} - \frac{B_2}{4} + \frac{B_3}{6} - \frac{B_4}{8} + \ldots.$$

Also solved by the Proposer.

MECHANICS.

227. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Regarding the earth as a homogeneous sphere, radius R, acceleration at the surface g, investigate the motion of a sphere, radius b, moving through a straight tunnel between two points on the surface not diametrically opposite.

II. Solution by the PROPOSER.

Let AB be the tunnel; C its mid-point; D the position of the sphere at any time t; O the center of the earth; $OD=y$, $AC=a$, $DC=x$, $\angle DOC=\theta$; f' the acceleration at D. Then $f' : y = g : R$ or $f' = gy/R$. But $y = x/\sin\theta$; also $f = f'\sin\theta$. $\therefore f = gx/R$.

$$\text{Now} \int_{v_1}^{v} v\,dv = f\int_{x}^{a} dx = \frac{g}{R}\int_{x}^{a} x\,dx.$$

$$\therefore v^2 = (dx/dt)^2 = g/R(a^2-x^2) + v_1^2.$$

$$\therefore t = \surd(R/g)\int_0^a \frac{dx}{\surd[Rv_1^2 + g(a^2-x^2))}.$$

$$\therefore t = (R/g)\sin^{-1}\frac{a\surd g}{\surd[Rv_1^2+ga^2]}.$$

If the sphere starts from rest, $v_1=0$.
$\therefore t = (\pi/2)\surd(R/g)$.

If the tunnel is perfectly rough and F is the friction, we get for the equations of motion: $d^2x/dt^2 + gx/R + F = 0$; $k^2(d^2\phi/dt^2) = bF$; $bd\phi = dx$.